

## CHAPTER 8

### BINOMIAL THEOREM

Binomial theorem for any positive integer n

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots + {}^nC_n b^n$$

Recall

$$1) {}^nC_r = \frac{n!}{(n-r)! r!}$$

$$2) {}^nC_r = {}^nC_{n-r}$$
$${}^7C_4 = {}^7C_3 = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$$

$${}^8C_6 = {}^8C_2 = \frac{8 \times 7}{1 \times 2} = 28$$

$$3) {}^nC_n = {}^nC_0 = 1$$

$$4) {}^nC_1 = n$$

#### OBSERVATIONS/ FORMULAS

- 1) The coefficients  ${}^nC_r$  occurring in the binomial theorem are known as binomial coefficients.
- 2) There are  $(n+1)$  terms in the expansion of  $(a+b)^n$ , ie one more than the index.
- 3) The coefficient of the terms equidistant from the beginning and end are equal.
- 4)  $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \dots + {}^nC_n x^n$ . (By putting  $a = 1$  and  $b = x$  in the expansion of  $(a + b)^n$ ).
- 5)  $(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots + (-1)^n {}^nC_n x^n$  (By putting  $a = 1$  and  $b = -x$  in the expansion of  $(a + b)^n$ ).
- 6)  $2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$  (By putting  $x = 1$  in (4))
- 7)  $0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n {}^nC_n$ . (By putting  $x = 1$  in (5))

8\*\*)  $(r + 1)^{\text{th}}$  term in the binomial expansion for  $(a+b)^n$  is called the general term which is given by

$$T_{r+1} = {}^nC_r a^{n-r} b^r.$$

i.e to find 4<sup>th</sup> term =  $T_4$ , substitute  $r = 3$ .

9\*) **Middle term** in the expansion of  $(a+b)^n$

i) If **n is even**, middle term =  $\left[\frac{n}{2} + 1\right]^{th}$  term.

ii) If **n is odd**, then 2 middle terms are,  $\left[\frac{n+1}{2}\right]^{th}$  term and  $\left[\frac{n+1}{2} + 1\right]^{th}$  term.

10\*) To find the **term independent of x or the constant term**, find the coefficient of  $x^0$ . (ie put power of  $x = 0$  and find  $r$ )

### Problems

eg 4\*\* (4 marks)

### Ex 8.1

Q 2,4,7,9 (1 mark)

10\*, 11\*, 12\* (4 marks)

13\*\*, 14\*\* (4 marks)

13\*\*) Show that  $9^{n+1} - 8n - 9$  is divisible by 64, whenever  $n$  is a positive integer

Or

$3^{2n+2} - 8n - 9$  is divisible by 64

Solution:  $9^{n+1} - 8n - 9 = (1+8)^{n+1} - 8n - 9$

$$= {}^{n+1}C_0 + {}^{n+1}C_1 8 + {}^{n+1}C_2 8^2 + {}^{n+1}C_3 8^3 + \dots + {}^{n+1}C_{n+1} 8^{n+1} - 8n - 9$$

$$= 1 + 8n + 8 + 8^2 [{}^{n+1}C_2 + {}^{n+1}C_3 \cdot 8 + \dots + 8^{n-1}] - 8n - 9$$

$$(\text{since } {}^{n+1}C_0 = {}^{n+1}C_{n+1} = 1, {}^{n+1}C_1 = {}^{n+1},$$

$$8^{n+1}/8^2 = 8^{n+1-2} = 8^{n-1})$$

$$= 8^2 [{}^{n+1}C_2 + {}^{n+1}C_3 \cdot 8 + \dots + 8^{n-1}] \text{ which is divisible by 64}$$

### Problems

eg 5\*, 6\*\*, 7\* (4 marks)

eg 8\*\*, 9\*\* (6 marks)

## Ex 8.2

Q 2,3\* (1 mark)

Q 7\*\*,8\*\*,9\*\*,11\*\*,12\*\* (4 marks), 10\*\* (6 marks)

eg 10\*\*,11 (HOT),12 (HOT), 13(HOT), eg 15\*,17\*\* (4 marks)

## Misc ex

Q 1\*\* (6 mark),2,3(HOT), 8\* (4 marks)

## Ex 8.2

Q 10\*\*(6 marks)

The coefficients of the  $(r-1)^{\text{th}}$ ,  $r^{\text{th}}$  and  $(r+1)^{\text{th}}$  terms in the expansion of  $(x+1)^n$  are in the ratio 1: 3 : 5. Find n and r.

Solution

$$T_{r+1} = {}^nC_r x^{n-r}$$

$$T_r = T_{(r-1)+1} = {}^nC_{r-1} x^{n-r+1}$$

$$T_{r-1} = T_{(r-2)+1} = {}^nC_{r-2} x^{n-r+2}$$

$$\text{Given } {}^nC_{r-2} : {}^nC_{r-1} : {}^nC_r :: 1 : 3 : 5$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3}$$

$$\frac{n!}{(n-r+2)!(r-2)!} \div \frac{n!}{(n-r+1)!(r-1)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+2)!} \times \frac{(r-1)!}{(r-2)!} = \frac{1}{3}$$

$$\frac{(n-r+1)!}{(n-r+1)!(n-r+2)} \times \frac{(r-2)!(r-1)}{(r-2)!} = \frac{1}{3}$$

$$\frac{r-1}{n-r+2} = \frac{1}{3}$$

$$3r-3 = n-r+2$$

$$n-4r = -5 \quad (1)$$

$$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{3}{5}$$

simplify as above and get the equation  $3n - 8r = -3$  \_\_\_\_\_(2)

solving (1) and (2) we get

$n = 7$  and  $r = 3$ .

### EXTRA/HOT QUESTIONS

- 1) Using Binomial theorem show that  $2^{3n} - 7n - 1$  or  $8^n - 7n - 1$  is divisible by 49 where  $n$  is a natural number. (4 marks\*\*)
- 2) Find the coefficient of  $x^3$  in the equation of  $(1+2x)^6 (1-x)^7$  (HOT)
- 3) Find  $n$  if the coefficient of  $5^{\text{th}}$ ,  $6^{\text{th}}$  &  $7^{\text{th}}$  terms in the expansion of  $(1+x)^n$  are in A.P.
- 4) If the coefficient of  $x^{r-1}$ ,  $x^r$ ,  $x^{r+1}$  in the expansion of  $(1+x)^n$  are in A.P. prove that  $n^2 - (4r+1)n + 4r^2 - 2 = 0$ . (HOT)
- 5) If  $6^{\text{th}}$ ,  $7^{\text{th}}$ ,  $8^{\text{th}}$  &  $9^{\text{th}}$  terms in the expansion of  $(x+y)^n$  are respectively  $a, b, c$  &  $d$  then show that  $\frac{b^2 - ac}{c^2 - bd} = \frac{4a}{3c}$  (HOT)
- 6) Find the term independent of  $x$  in the expansion of  $\left[3x^2 - \frac{1}{2x^3}\right]^{10}$  (4 marks\*)
- 7) Using Binomial theorem show that  $3^{3n} - 26n - 1$  is divisible by 676. (4 marks\*\*)
- 8) The  $3^{\text{rd}}$ ,  $4^{\text{th}}$  &  $5^{\text{th}}$  terms in the expansion of  $(x+a)^n$  are 84, 280 & 560 respectively. Find the values of  $x$ ,  $a$  and  $n$ . (6 marks\*\*)
- 9) The coefficient of 3 consecutive terms in the expansion of  $(1+x)^n$  are in the ratio 3 : 8 : 14. Find  $n$ . (6 mark\*\*)
- 10) Find the constant term in the expansion of  $(x-1/x)^{14}$
- 11) Find the middle term(s) in the expansion of
  - i)  $\left[\frac{x}{a} - \frac{a}{x}\right]^{10}$
  - ii)  $\left[2x - \frac{x^2}{4}\right]^9$
- 12) If  $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$   
Prove that  $C_1 + 2C_2 + 3C_3 + \dots + nC_n = n \cdot 2^{n-1}$

### Answers

2) -43

3)  $n = 7$  or 14

6)  $76545/8$

8)  $x = 1$ ,  $a = 2$ ,  $n = 7$

9) 10

10) -3432

11) i) -252

ii)  $-\frac{63}{32} x^{14}$

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